Random Walks on Groups

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ABSTRACT

A finite group G is partitioned into nonempty disjoint subsets $C_0, C_1, \dots, C_m$ such that for every $g_1 \in C_i$ the number of ordered pairs (g_2, g_3) for which $g_2 \in C_\nu$, $g_3 \in C_j$, and $g_1 g_2 = g_3$ is independent of the particular choice of g_1. A sequence of mutually independent random elements $\gamma_0, \gamma_1, \dots, \gamma_n, \dots$ is chosen in G in such a way that for $n = 1, 2, \dots$ the probability $\mathbf{P}\{\gamma_n = g\}$ depends only on the class C_ν which contains g. Let $\xi_n = j$ if $\gamma_0 \gamma_1 \cdots \gamma_n \in C_j$. Then $\{\xi_n;\ n \geqq 0\}$ is a homogeneous Markov chain with state space $I = \{0, 1, \dots, m\}$. The aim of this paper is to determine the n-step transition probabilities of the Markov chain $\{\xi_n;\ n \geqq 0\}$. The results derived in this paper lead also to a probabilistic interpretation and a generalization of group characters.

1. INTRODUCTION

This paper is concerned with a problem of random walks on finite groups. To define such a random walk let us consider a sequence of mutually independent random elements $\gamma_0, \gamma_1, \dots, \gamma_n, \dots$ each belonging to a finite group G of order w. The unit element of G is denoted by e. The sequence of products $\gamma_0 \gamma_1 \cdots \gamma_n$ $(n = 0, 1, 2, \dots)$ forms a Markov chain and describes a random walk on the group G. Most of the recurrence properties of this random walk can be deduced from the probabilities

$$p(n) = \mathbf{P}\{\gamma_0 \gamma_1 \cdots \gamma_n \in C_0 \mid \gamma_0 = e\} \qquad (n \geqq 0) \tag{1}$$

where C_0 is a specific subset of G. The main problem of this paper is concerned with the determination of $p(n)$ for all $n \geqslant 0$. In the above formulation, the solution of the problem of finding $p(n)$ is too general to have any practical value. The most we can say is that $p(n)$ can be determined by

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calculating the n-step transition probabilities for a Markov chain with w states. In order that we can deduce some useful formulas for $p(n)$, we have to impose some restrictions on the probability distributions $\mathbf{P}\{\gamma_n=g\}$ ($g\in G$, $n\geqslant 0$). Our intention is to impose only some minimal restrictions so that we can include all the interesting random walks on G.

To furnish some appropriate restrictions on $\mathbf{P}\{\gamma_n=g\}$ ($g\in G$) let us suppose that G is partitioned into $m+1$ disjoint nonempty subsets $C_0, C_1,\ldots, C_m$, where C_0 contains e. Denote by w_i the number of elements of C_i, and by $I=\{0,1,\ldots,m\}$ the index set of the partition. Let us suppose that the random element γ_0 has an arbitrary distribution on G, whereas the random elements $\gamma_1, \gamma_2,\ldots,\gamma_n,\ldots$ are identically distributed and the probability $\mathbf{P}\{\gamma_n=g\}$ does not depend on the particular g, but only on the class C_ν ($\nu=0,1,\ldots,m$) which contains g. We write

$$\mathbf{P}\{\gamma_n=g\}=p_\nu/w_0 \tag{2}$$

for $n\geqslant 1$ and $g\in C_\nu$. The sum of the probabilities (2) for all $g\in G$ is necessarily equal to 1, that is,

$$\sum_{\nu=0}^{m} w_\nu p_\nu = w_0. \tag{3}$$

It is also assumed that the partition $\{C_0, C_1,\ldots, C_m\}$ satisfies the following condition.

Condition 1. For $i, j, \nu\in I$ and $g_1\in C_i$, the number of ordered pairs (g_2, g_3) for which $g_2\in C_\nu$, $g_3\in C_j$, and $g_1g_2=g_3$ is independent of the particular choice of g_1, and is equal to $w_0 a_{ij\nu}$, where $a_{ij\nu}$ is a nonnegative rational number.

Now let us define a sequence of discrete random variables $\xi_0, \xi_1,\ldots,\xi_n,\ldots$ such that

$$\xi_n=j \qquad \text{if} \quad \gamma_0\gamma_1\cdots\gamma_n\in C_j. \tag{4}$$

Then $\{\xi_n;\ n\geqslant 0\}$ describes a random walk on the partition $\{C_0, C_1,\ldots, C_m\}$, and Condition 1 implies that $\{\xi_n;\ n\geqslant 0\}$ is a homogeneous Markov chain with state space $I=\{0,1,\ldots,m\}$ and transition probability matrix

$$\boldsymbol{\pi} = \sum_{\nu=0}^{m} p_\nu \mathbf{A}_\nu, \tag{5}$$

where

$$\mathbf{A}_\nu = [a_{ij\nu}]_{i,j\in I} \tag{6}$$

for $\nu = 0, 1, \ldots, m$.

In terms of the Markov chain $\{\xi_n;\ n \geqslant 0\}$ we can express (1) in the following form:

$$p(n) = \mathbf{P}\{\xi_n = 0 | \xi_0 = 0\} \tag{7}$$

for $n \geqslant 0$.

If we form the nth power of the matrix $\boldsymbol{\pi}$, then we obtain a matrix whose elements, $p_{ik}^{(n)}$ $(i, k \in I)$, are the n-step transition probabilites of the Markov chain $\{\xi_n;\ n \geqslant 0\}$, that is,

$$\boldsymbol{\pi}^n = \left[p_{ik}^{(n)}\right]_{i,k\in I}, \tag{8}$$

and

$$p(n) = p_{00}^{(n)} \tag{9}$$

yields the desired probability (1). Thus the problem of finding $p(n)$ can be reduced to the problem of finding $\boldsymbol{\pi}^n$ for $n \geqslant 0$. If m is much smaller than w, then it is a great advantage to use $\{\xi_n;\ n \geqslant 0\}$ instead of $\{\gamma_0\gamma_1 \cdots \gamma_n;\ n \geqslant 0\}$ for the determination of $p(n)$.

The elements of the matrix $\boldsymbol{\pi}$ depend on the parameters $p_0, p_1, \ldots, p_m$, and at first sight it seems a very difficult task to determine $\boldsymbol{\pi}^n$ for all $n \geqslant 0$ and for all possible choices of the parameters $p_0, p_1, \ldots, p_m$. However, if the partition $\{C_0, C_1, \ldots, C_m\}$ satisfies also Condition 2 formulated below, then we can deduce a general formula for $\boldsymbol{\pi}^n$ from the properties of the matrices $\mathbf{A}_0, \mathbf{A}_1, \ldots, \mathbf{A}_m$ and the result is valid for any choice of the parameters $p_0, p_1, \ldots, p_m$.

Condition 2. For any $\nu \in I$ there is a $\nu' \in I$ such that $g \in C_\nu$ implies that $g^{-1} \in C_{\nu'}$.

In what follows we shall give several examples for partitions of finite groups satisfying Conditions 1 and 2. Then we shall study the properties of the matrices $\mathbf{A}_0, \mathbf{A}_1, \ldots, \mathbf{A}_m$, and determine $\boldsymbol{\pi}^n$ for all $n \geqslant 0$ and for any choice of the parameters $p_0, p_1, \ldots, p_m$. We shall point out that these results lead to a generalization of group characters. All these discussions indicate that the notion of group characters can be introduced in a simple way by considering random walks on groups.

2. RANDOM WALK ON A REGULAR POLYTOPE

Let us suppose that G is a finite group and H is a subgroup of G. For each $g \in G$ let us form the class

$$C(g) = HgH = \{h_1 g h_2 : h_1 \in H \text{ and } h_2 \in H\}. \tag{10}$$

Any two classes $C(g_1)$ and $C(g_2)$ are either disjoint or identical. Denote by $C_0, C_1, \ldots, C_m$ all the disjoint classes of type (10). Then the partition $\{C_0, C_1, \ldots, C_m\}$ satisfies Conditions 1 and 2, and $\{\xi_n;\, n \geqslant 0\}$ defined by (4) is a homogeneous Markov chain with state space $I = \{0, 1, \ldots, m\}$. In this case $C_0 = H$, w_0 is the order of H, and $a_{ij\nu}$ $(i, j, \nu \in I)$ are nonnegative integers.

An interesting particular case of the above example is concerned with regular polytopes. For the theory of regular polytopes we refer to H. S. M. Coxeter [3]. Let $\mathfrak{P}$ be a regular polytope with σ vertices. Denote by $\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_{\sigma-1}$ the rectangular Cartesian coordinates of the vertices of $\mathfrak{P}$ in a coordinate system whose origin is in the center of $\mathfrak{P}$. Let G be the symmetry group of $\mathfrak{P}$, and H be the maximal subgroup of G which leaves a given vertex, say $\mathbf{x}_0$, fixed. Let us define a partition $\{C_0, C_1, \ldots, C_m\}$ of G by (10), and define $\{\xi_n;\, n \geqslant 0\}$ by (4). We write $\mathbf{x}_r g = \mathbf{x}_s$ if $g \in G$ maps the vertex $\mathbf{x}_r$ into $\mathbf{x}_s$. We divide the vertices of $\mathfrak{P}$ into sections $S_0, S_1, \ldots, S_m$ defined by

$$S_j = \{\mathbf{x}_s : \mathbf{x}_0 g = \mathbf{x}_s \text{ and } g \in C_j\} \tag{11}$$

for $j = 0, 1, \ldots, m$. Clearly, $S_0 = \{\mathbf{x}_0\}$. Denote by σ_j the number of vertices in the set S_j. We have $\sigma_0 = 1$ and $w_j = \sigma_j w_0$ for $j = 0, 1, \ldots, m$. In this case every $a_{ij\nu}$ $(i, j, \nu \in I)$ is a nonnegative integer.

Let us consider a random walk on $\mathfrak{P}$ in which in a series of random steps a traveler visits the vertices of the polytope. The traveler starts at a given vertex and in each step, independently of the past journey, chooses a vertex at random as the destination. Denote by $\mathbf{v}_n$ $(n = 1, 2, \ldots)$ the position of the traveler at the end of the nth step, and by $\mathbf{v}_0$ the initial position. It is assumed that

$$\mathbf{v}_n = \mathbf{v}_{n-1} \gamma_n \tag{12}$$

for $n = 1, 2, \ldots,$ and $\mathbf{v}_0 = \mathbf{x}_0 \gamma_0$, where $\mathbf{x}_0$ is a given vertex of $\mathfrak{P}$, and

$\gamma_0, \gamma_1, \ldots, \gamma_n, \ldots$ is a sequence of independent random elements each belonging to G. For $n \geq 1$ the distribution of γ_n is assumed to be given by (2). Then (3) reduces to

$$\sum_{\nu=0}^{m} \sigma_\nu p_\nu = 1. \tag{13}$$

Now we can express (4) in the following equivalent form:

$$\xi_n = j \qquad \text{if} \quad \mathbf{v}_n \in S_j. \tag{14}$$

The probability $p(n)$ can be interpreted as the probability that the traveler returns to the initial position at the end of the nth step.

The model described above contains several important particular cases. To consider these cases let us introduce two distance functions on the vertices of $\mathfrak{P}$. We define $D(\mathbf{x}_r, \mathbf{x}_s)$ as the minimal number of edges in the paths connecting $\mathbf{x}_r$ and $\mathbf{x}_s$, and $d(\mathbf{x}_r, \mathbf{x}_s)$ as the Euclidean distance $\|\mathbf{x}_r - \mathbf{x}_s\|$ between $\mathbf{x}_r$ and $\mathbf{x}_s$. Denote by $0, 1, \ldots, l$ the possible values of $D(\mathbf{x}_r, \mathbf{x}_s)$, and by $d_0, d_1, \ldots, d_\mu$ the possible values of $d(\mathbf{x}_r, \mathbf{x}_s)$ arranged in increasing order of magnitude. Clearly, $d_0 = 0$. If, as an alternative to (12), we assume that the transition probabilities $\mathbf{P}\{\mathbf{v}_n = \mathbf{x}_s | \mathbf{v}_{n-1} = \mathbf{x}_r\}$ $(r, s = 0, 1, \ldots, \sigma - 1)$ depend either only on $D(\mathbf{x}_r, \mathbf{x}_s)$, or only on $d(\mathbf{x}_r, \mathbf{x}_s)$, then in fact we are considering two particular cases of (12). For each vertex $\mathbf{x}_s$ in S_j has the same distance $D(\mathbf{x}_0, \mathbf{x}_s)$ from $\mathbf{x}_0$, and also has the same distance $d(\mathbf{x}_0, \mathbf{x}_s)$ from $\mathbf{x}_0$. A particularly important case is the following:

$$\mathbf{P}\{\mathbf{v}_n = \mathbf{x}_s | \mathbf{v}_{n-1} = \mathbf{x}_r\} = \begin{cases} 1/q & \text{if} \quad D(\mathbf{x}_r, \mathbf{x}_s) = 1, \\ 0 & \text{otherwise,} \end{cases} \tag{15}$$

where q is the number of edges emanating from each vertex of $\mathfrak{P}$. In this case in each step the traveler moves to one of the neighboring vertices with a constant probability.

Table 1 contains all the regular polytopes and the quantities σ, q, w_0, m, μ, and l. For each regular polytope $\mathfrak{P}$ the probability $p(n)$ $(n \geq 0)$ is given in Reference [9].

Finally, we note that the sections S_j $(j = 0, 1, \ldots, m)$ can also be characterized in the following way: If $\mathfrak{P}$ is any regular polytope other than the four-dimensional 24-cell, 600-cell, and 120-cell, then S_j contains all those vertices $\mathbf{x}_s$ of $\mathfrak{P}$ for which $D(\mathbf{x}_0, \mathbf{x}_s) = j$ or, equivalently, $d(\mathbf{x}_0, \mathbf{x}_s) = d_j$. If $\mathfrak{P}$ is the four-dimensional 24-cell or 600-cell, then S_j contains all those vertices $\mathbf{x}_s$ of $\mathfrak{P}$ for which $d(\mathbf{x}_0, \mathbf{x}_s) = d_j$. If $\mathfrak{P}$ is the four-dimensional 120-cell, then we should use (11) in characterizing the sections S_j $(j = 0, 1, \ldots, 44)$ of $\mathfrak{P}$.

TABLE 1

Dimensions	Regular polytope	σ	q	w_0	m	μ	l
2	p-gon	p	2	2	$[p/2]$	$[p/2]$	$[p/2]$
3	Tetrahedron	4	3	6	1	1	1
	Octahedron	6	4	8	2	2	2
	Cube	8	3	6	3	3	3
	Icosahedron	12	5	10	3	3	3
	Dodecahedron	20	3	6	5	5	5
4	5-cell	5	4	24	1	1	1
	16-cell	8	6	48	2	2	2
	8-cell	16	4	24	4	4	4
	24-cell	24	8	48	4	4	3
	600-cell	120	12	120	8	8	5
	120-cell	600	4	24	44	30	15
$r\geq 5$	Regular simplex	$r+1$	r	$r!$	1	1	1
	Cross polytope	$2r$	$2(r-1)$	$2^{r-1}(r-1)!$	2	2	2
	Measure polytope	2^r	r	$r!$	r	r	r

3. GROUP CHARACTERS

Let us suppose again that G is a finite group and H is a subgroup of G. For each $g\in G$ let us form the class

$$C(g)=\{hgh^{-1}: h\in H\}. \tag{16}$$

Any two classes $C(g_1)$ and $C(g_2)$ are either disjoint or identical. Denote by $C_0, C_1,\dots, C_m$ all the disjoint classes of type (16). Then the partition $\{C_0, C_1,\dots, C_m\}$ satisfies Conditions 1 and 2, and $\{\xi_n;\ n\geq 0\}$ defined by (4) is a homogeneous Markov chain with state space $I=\{0,1,\dots,m\}$. In this case $C_0=\{e\}$, $w_0=1$, and $a_{ij\nu}$ $(i,j,\nu\in I)$ are nonnegative integers.

If, in particular, $H=G$, then $C_0, C_1,\dots, C_m$ are the conjugacy classes of G and the problem of finding $p(n)$ leads in a natural way to the definition of group characters. It is interesting to note that F. G. Frobenius [5] in 1896 discovered the notion of group characters by studying group determinants. As an alternative we arrive at the notion of group characters by studying random walks on finite groups. This latter approach leads not only to the notion of group characters, but also to a generalization of it.

4. PARTITIONS OF GROUPS

In this section we shall prove some general theorems for a partition of a finite group in the case where Conditions 1 and 2 are satisfied.

Let G be a finite group of order w. Let us suppose that G is partitioned into $m+1$ disjoint nonempty subsets $C_0, C_1, \ldots, C_m$ where C_0 contains e, the unit element of G. Denote by w_i the number of elements of C_i, and write $\sigma_i = w_i/w_0$ for $i=0,1,\ldots,m$ and $\sigma = w/w_0$. Then

$$\sum_{i=0}^{m} \sigma_i = \sigma \tag{17}$$

and $\sigma_0 = 1$.

Throughout this section we assume that the partition $\{C_0, C_1, \ldots, C_m\}$ satisfies Conditions 1 and 2 and that $\mathbf{A}_\nu$ $(\nu=0,1,\ldots,m)$ are defined by (6).

We shall use the Kronecker symbol defined by

$$\delta_{ij} = \begin{cases} 1 & \text{if} \quad i=j, \\ 0 & \text{if} \quad i\neq j, \end{cases} \tag{18}$$

and write $I = \{0,1,\ldots,m\}$.

Obviously, we have $\mathbf{A}_0 = \mathbf{I}$ where $\mathbf{I} = [\delta_{ij}]_{i,j\in I}$ is the $(m+1)\times(m+1)$ unit matrix, and $a_{0j\nu} = \sigma_j \delta_{j\nu}$. In the matrix $\mathbf{A}_\nu$ each row sum is equal to σ_ν, that is,

$$\sum_{j=0}^{m} a_{ij\nu} = \sigma_\nu \tag{19}$$

for any $i \in I$ and $\nu \in I$.

We observe that if $\nu' = \nu$, then C_ν contains the inverse of each of its elements. If $\nu' \neq \nu$, then $C_{\nu'}$ consists of the inverses of the elements of C_ν. Obviously, $\sigma_{\nu'} = \sigma_\nu$ for all $\nu \in I$. The integers $0', 1', \ldots, m'$ form a permutation of $0,1,\ldots,m$. We define the corresponding permutation matrix $\boldsymbol{\Delta}$ by

$$\boldsymbol{\Delta} = \left[\delta_{ij'}\right]_{i,j\in I}. \tag{20}$$

We have $\boldsymbol{\Delta}' = \boldsymbol{\Delta}$, where $\boldsymbol{\Delta}'$ is the transpose of $\boldsymbol{\Delta}$, and $\boldsymbol{\Delta}^2 = \mathbf{I}$, where $\mathbf{I}$ is the $(m+1)\times(m+1)$ unit matrix.

The next few theorems characterize certain properties of the elements of the matrices $\mathbf{A}_0, \mathbf{A}_1, \ldots, \mathbf{A}_m$.

THEOREM 1. *The numbers $a_{ij\nu}$ $(i, j, \nu \in I)$ defined in Condition 1 satisfy the following equations*:

$$\sigma_i a_{ij\nu} = \sigma_j a_{ji\nu'} \tag{21}$$

and

$$a_{ij\nu} = a_{i'\nu j}. \tag{22}$$

Proof. By Condition 1 the number of triplets (g_1, g_2, g_3) satisfying the requirements of $g_1 \in C_i$, $g_2 \in C_\nu$, $g_3 \in C_j$, and $g_1 g_2 = g_3$ is $w_0^2 \sigma_i a_{ij\nu}$. Since $g_1 g_2 = g_3$ if and only if $g_3 g_2^{-1} = g_1$ or $g_1^{-1} g_3 = g_2$, and since now $g_2^{-1} \in C_{\nu'}$ and $g_1^{-1} \in C_{i'}$, therefore we have

$$\sigma_i a_{ij\nu} = \sigma_j a_{ji\nu'} = \sigma_i a_{i'\nu j}. \tag{23}$$

This proves (21) and (22). ∎

Equations (21) and (22) can conveniently be expressed in matrix notation. Let us introduce the diagonal matrix

$$\mathbf{D} = \left[\delta_{ij} \sigma_j^{1/2}\right]_{i,j \in I}, \tag{24}$$

where the square root is positive. By (21) we have

$$\mathbf{D}^2 \mathbf{A}_\nu = \mathbf{A}'_{\nu'} \mathbf{D}^2 \tag{25}$$

for $\nu \in I$, where the prime means transposition.

By (22) we obtain that

$$[a_{ij\nu}]_{i,\nu \in I} = \boldsymbol{\Delta} [a_{i\nu j}]_{i,\nu \in I} = \boldsymbol{\Delta} \mathbf{A}_j \tag{26}$$

for $j \in I$, where $\boldsymbol{\Delta}$ is the permutation matrix defined by (20).

By (21) and (22) we have $\sigma_i a_{ij\nu} = \sigma_j a_{j'\nu' i}$, and this implies that

$$\sigma_i [a_{ij\nu}]_{j,\nu \in I} = \mathbf{D}^2 \boldsymbol{\Delta} \mathbf{A}_i \boldsymbol{\Delta} \tag{27}$$

for $i \in I$.

THEOREM 2. *The matrices* $\mathbf{\Delta A}_j\mathbf{\Delta}$ *and* $\mathbf{A}_\nu$ *commute, that is,*

$$\mathbf{\Delta A}_j\mathbf{\Delta A}_\nu = \mathbf{A}_\nu\mathbf{\Delta A}_j\mathbf{\Delta} \tag{28}$$

for all $j \in I$ *and all* $\nu \in I$.

Proof. If C_i denotes also the sum of the elements of C_i, then by Condition 1 and by (21) we can write that

$$C_iC_\nu = w_0 \sum_{j=0}^{m} a_{ji\nu'}C_j \tag{29}$$

for any $i \in I$ and $\nu \in I$. If we use (29) repeatedly, then we get

$$(C_iC_\nu)C_k = w_0^2 \sum_{l=0}^{m} a_{li\nu'}C_lC_k = \sum_{j=0}^{m}\sum_{l=0}^{m} a_{li\nu'}a_{jlk'}C_j \tag{30}$$

and

$$C_i(C_\nu C_k) = w_0^2 \sum_{l=0}^{m} a_{l\nu k'}C_iC_l = \sum_{j=0}^{m}\sum_{l=0}^{m} a_{l\nu k'}a_{jil'}C_j. \tag{31}$$

Since

$$(C_iC_\nu)C_k = C_i(C_\nu C_k), \tag{32}$$

by comparing the coefficients of C_j in (30) and (31) we get the identity

$$\sum_{l=0}^{m} a_{li\nu'}a_{jlk'} = \sum_{l=0}^{m} a_{l\nu k'}a_{jil'}. \tag{33}$$

If we consider both sides of (33) as the (i, k)-entry of an $(m+1)\times(m+1)$ matrix, then we get the matrix equation

$$[a_{li\nu'}]_{i,l}[a_{jlk'}]_{l,k} = [a_{jil'}]_{i,l}[a_{l\nu k'}]_{l,k}. \tag{34}$$

In (34) the first matrix is

$$[a_{ji\nu'}]_{i,j} = \mathbf{A}'_{\nu'} = \mathbf{D}^2\mathbf{A}_\nu\mathbf{D}^{-2}, \tag{35}$$

the second and third matrices are

$$[a_{ji\nu'}]_{i,\nu} = \sigma_j^{-1}\mathbf{D}^2\boldsymbol{\Delta}\mathbf{A}_j, \tag{36}$$

and the fourth matrix is

$$[a_{i\nu j'}]_{i,j} = \boldsymbol{\Delta}\mathbf{A}_\nu\boldsymbol{\Delta}. \tag{37}$$

These formulas follow from (25), (27), and (26) respectively. Putting (35), (36), and (37) into (34), we get the identity

$$\mathbf{D}^2\mathbf{A}_\nu\boldsymbol{\Delta}\mathbf{A}_j = \mathbf{D}^2\boldsymbol{\Delta}\mathbf{A}_j\boldsymbol{\Delta}\mathbf{A}_\nu\boldsymbol{\Delta}, \tag{38}$$

which proves (28). ■

THEOREM 3. *If $\nu' = \nu$ for all $\nu \in I$, then the matrices $\mathbf{A}_0, \mathbf{A}_1, \ldots, \mathbf{A}_m$ commute in pairs, that is,*

$$\mathbf{A}_j\mathbf{A}_\nu = \mathbf{A}_\nu\mathbf{A}_j \tag{39}$$

for all $j \in I$ and all $\nu \in I$.

Proof. If $\nu' = \nu$ for all $\nu \in I$, then in (20), $\delta_{ij'} = \delta_{ij}$ and $\boldsymbol{\Delta}$ reduces to the unit matrix $\mathbf{I}$. Thus (39) follows from (28). ■

THEOREM 4. *If*

$$a_{ij\nu'} = a_{i\nu j'} \tag{40}$$

holds for all $i, j, \nu \in I$, then $\mathbf{A}_0, \mathbf{A}_1, \ldots, \mathbf{A}_m$ commute in pairs, that is, (39) holds for all $j \in I$ and all $\nu \in I$.

Proof. By (25) we can express (40) in the following equivalent form:

$$\mathbf{A}_{\nu'} = \boldsymbol{\Delta}\mathbf{A}_\nu\boldsymbol{\Delta} \tag{41}$$

for $\nu \in I$, where $\boldsymbol{\Delta}$ is defined by (20). If we make use of (41), then (28) reduces to the equation

$$\mathbf{A}_{j'}\mathbf{A}_{\nu}=\mathbf{A}_{\nu}\mathbf{A}_{j'} \tag{42}$$

which is valid for any $j\in I$ and $\nu\in I$. This proves (39). ■

We note that Equation (40) is equivalent to

$$C_jC_\nu=C_\nu C_j, \tag{43}$$

where $j\in I$ and $\nu\in I$.

5. THE DETERMINATION OF $\boldsymbol{\pi}^n$

In this section we shall use the same notation as in Section 4, and we assume again that the partition $\{C_0, C_1,\dots, C_m\}$ satisfies Conditions 1 and 2 and that $\mathbf{A}_\nu$ $(\nu=0,1,\dots,m)$ are defined by (6).

Our aim is to determine the nth power of the matrix

$$\boldsymbol{\pi}=\sum_{\nu=0}^{m} p_\nu \mathbf{A}_\nu, \tag{44}$$

where $p_0, p_1,\dots, p_m$ are arbitrary real or complex numbers. The following theorem provides a simple solution if the matrices $\mathbf{A}_0,\mathbf{A}_1,\dots,\mathbf{A}_m$ commute in pairs.

THEOREM 5. *If the partition $C_0, C_1,\dots, C_m$ satisfies Conditions 1 and 2 in Section 1 and if the matrices $\mathbf{A}_0,\mathbf{A}_1,\dots,\mathbf{A}_m$ commute in pairs, then there exists a nonsingular matrix*

$$\mathbf{H}=\left[h_{ij}\right]_{i,j\in I} \tag{45}$$

such that

$$\left[h_{ij}\sigma_i^{1/2}\right]_{i,j\in I} \tag{46}$$

is a unitary matrix, and

$$\mathbf{A}_\nu=\mathbf{H}\boldsymbol{\Lambda}_\nu\mathbf{H}^{-1} \tag{47}$$

for all $\nu \in I$, *where*

$$\mathbf{\Lambda}_\nu = \left[\delta_{ij}\lambda_{j\nu}\right]_{i,j\in I} \tag{48}$$

is a diagonal matrix whose diagonal elements

$$\lambda_{j\nu} = \sigma_\nu h_{\nu j}/h_{0j} \qquad (j\in I) \tag{49}$$

are the eigenvalues of $\mathbf{A}_\nu$.

If we choose h_{0j} $(j\in I)$ *to be positive real numbers, then the matrix* $\mathbf{H}$ *is uniquely determined up to a permutation of its columns.*

Proof. Since the matrices $\mathbf{A}_0, \mathbf{A}_1, \ldots, \mathbf{A}_m$ commute in pairs, therefore the matrices

$$(1-i)\mathbf{D}(\mathbf{A}_\nu + i\mathbf{A}_{\nu'})\mathbf{D}^{-1} \qquad (\nu = 0, 1, \ldots, m) \tag{50}$$

also commute in pairs. By (25) it follows that

$$\mathbf{D}\mathbf{A}_\nu\mathbf{D}^{-1} = \left(\mathbf{D}\mathbf{A}_{\nu'}\mathbf{D}^{-1}\right)', \tag{51}$$

where $\mathbf{D}$ is defined by (24). Consequently, the matrices (50) are Hermitian, and therefore their eigenvalues are real numbers. In what follows a bar means complex conjugate.

Since the matrices (50) commute in pairs, by a theorem of F. G. Frobenius [4] there exists a unitary matrix $\mathbf{U}$ for which

$$\mathbf{U}\overline{\mathbf{U}}' = \overline{\mathbf{U}}'\mathbf{U} = \mathbf{I} \tag{52}$$

such that

$$(1-i)\mathbf{D}(\mathbf{A}_\nu + i\mathbf{A}_{\nu'})\mathbf{D}^{-1} = \mathbf{U}\mathbf{L}_\nu\mathbf{U}^{-1} \tag{53}$$

for all $\nu \in I$, where $\mathbf{L}_\nu$ is a diagonal matrix whose diagonal elements are real numbers. We note that $\mathbf{U}$ is not unique in (53). If we multiply each column of $\mathbf{U}$ by a complex number having absolute value 1, then (53) holds unchangeably. If in (53) we form a permutation of the columns of $\mathbf{U}$, then it changes only the order of the diagonal elements of $\mathbf{L}_\nu$.

If we interchange ν and ν' in (53) and multiply both sides by $-i$, then we get

$$(1-i)\mathbf{D}(\mathbf{A}_\nu - i\mathbf{A}_{\nu'})\mathbf{D}^{-1} = -i\mathbf{U}\mathbf{L}_{\nu'}\mathbf{U}^{-1}. \tag{54}$$

By adding (53) and (54) we obtain

$$\mathbf{D}\mathbf{A}_\nu\mathbf{D}^{-1} = \mathbf{U}\mathbf{\Lambda}_\nu\mathbf{U}^{-1}, \tag{55}$$

where

$$\mathbf{\Lambda}_\nu = \left[\delta_{jk}\lambda_{k\nu}\right]_{j,k\in I} = \tfrac{1}{4}(\mathbf{L}_\nu + \mathbf{L}_{\nu'}) + \frac{i}{4}(\mathbf{L}_\nu - \mathbf{L}_{\nu'}) \tag{56}$$

is a diagonal matrix whose diagonal elements are the eigenvalues of $\mathbf{A}_\nu$.

The matrix

$$\mathbf{H} = \mathbf{D}^{-1}\mathbf{U} = \left[h_{ij}\right]_{i,j\in I} \tag{57}$$

is nonsingular, and its inverse is

$$\mathbf{H}^{-1} = \overline{\mathbf{H}}'\mathbf{D}^2 = \left[\bar{h}_{ji}\sigma_j\right]_{i,j\in I}. \tag{58}$$

By (55)

$$\mathbf{A}_\nu = \mathbf{H}\mathbf{\Lambda}_\nu\mathbf{H}^{-1} \tag{59}$$

is a Jordan decomposition of $\mathbf{A}_\nu$. This is in agreement with (47). Since

$$\mathbf{U} = \mathbf{D}\mathbf{H} = \left[h_{ij}\sigma_i^{1/2}\right]_{i,j\in I}, \tag{60}$$

the matrix (46) is a unitary matrix.

Now we shall show that if h_{0j} $(j\in I)$ are chosen to be positive real numbers, then $\mathbf{H}$ is unique up to a permutation of its columns. Let us write (59) in the form

$$\mathbf{A}_\nu\mathbf{H} = \mathbf{H}\mathbf{\Lambda}_\nu \tag{61}$$

and form the $(0, j)$-entry of both sides. Then we get the equation

$$\sigma_\nu h_{\nu j} = h_{0j}\lambda_{j\nu} \tag{62}$$

for $j \in I$ and $\nu \in I$. In (62) $\sigma_\nu \neq 0$, and since $\mathbf{H}$ is nonsingular, $h_{\nu j} \neq 0$ for some $\nu \in I$. Therefore $h_{0j} \neq 0$ for all $j \in I$. If $h_{0j} \neq 0$ is known, then by (62) the eigenvalues of $\mathbf{A}_\nu$ uniquely determine $h_{\nu j}$ for $\nu \in I$. We cannot choose h_{0j} in an arbitrary way. Since (60) is a unitary matrix, we have $\overline{\mathbf{U}}'\mathbf{U} = \mathbf{I}$, and if we form the (j, j)-entry in this matrix, we get

$$|h_{0j}|^2 \sigma_j \sum_{\nu=0}^{m} |\lambda_{j\nu}|^2 \sigma_\nu^{-2} = 1 \tag{63}$$

for $\nu \in I$. If in (53) we multiply each column of $\mathbf{U}$ by a properly chosen complex number having absolute value 1, then we can achieve that h_{0j} becomes a positive real number for every $j \in I$. Then h_{0j} is uniquely determined by (63), and $h_{\nu j}$ $(\nu \in I, j \in I)$ is uniquely determined by (62). Of course, $h_{\nu j}$ $(\nu \in I,\ j \in I)$ depends on the order in which the eigenvalues of the matrices $\mathbf{A}_\nu$ $(\nu \in I)$ are arranged. The matrix $\mathbf{H}$ is uniquely determined by the eigenvalues of $\mathbf{A}_\nu$ $(\nu \in I)$, if these eigenvalues are arranged in a specific order. Conversely, if $\mathbf{H}$ is known, then the eigenvalues of $\mathbf{A}_\nu$ $(\nu \in I)$ are uniquely determined by (62). The formula (62) proves (49), and this completes the proof of the theorem. ■

In the proof of Theorem 5 we demonstrated that if the matrices $\mathbf{A}_0, \mathbf{A}_1, \ldots, \mathbf{A}_m$ commute in pairs, then they determine a unique matrix $\mathbf{H}$. Conversely, if $\mathbf{H}$ is known, then the matrices $\mathbf{A}_0, \mathbf{A}_1, \ldots, \mathbf{A}_m$ are uniquely determined by $\mathbf{H}$. This is the content of the next theorem.

THEOREM 6. *If the conditions of Theorem 5 are satisfied, then the elements of the matrix $\mathbf{A}_\nu$ $(\nu \in I)$ can be expressed in the following form:*

$$a_{ik\nu} = \sigma_k \sigma_\nu \sum_{j=0}^{m} \frac{h_{ij} \bar{h}_{kj} h_{\nu j}}{h_{0j}}, \tag{64}$$

where $i, k, \nu \in I$ and the bar means complex conjugate.

Proof. By (56), (57), (58), and (59) we have

$$\mathbf{A}_\nu = \left[h_{ij}\right]_{i,j \in I} \left[\delta_{jk} \lambda_{k\nu}\right]_{j,k \in I} \left[\bar{h}_{ji} \sigma_i\right]_{i,j \in I}. \tag{65}$$

If we form the (i, k)-entry of (65) and if we use (62), then we get (64). ■

As we have demonstrated, the matrix $\mathbf{H}$ contains all the information concerning $\mathbf{A}_0, \mathbf{A}_1, \ldots, \mathbf{A}_m$.

It remains to determine the elements of the matrix

$$\pi^n = \left[p_{ik}^{(n)}\right], \tag{66}$$

where π is defined by (44). We shall prove the following result.

THEOREM 7. *If the conditions of Theorem 5 are satisfied, then the elements of the matrix* π^n $(n \geq 0)$ *can be expressed in the following form*:

$$p_{ik}^{(n)} = \sigma_k \sum_{j=0}^{m} h_{ij} \bar{h}_{kj} \lambda_j^n, \tag{67}$$

where $i \in I$, $k \in I$, $n \geq 0$, *and*

$$\lambda_j = \sum_{\nu=0}^{m} \frac{p_\nu \sigma_\nu h_{\nu j}}{h_{0j}} \qquad (j=0,1,\ldots,m) \tag{68}$$

are the eigenvalues of π.

Proof. From (44) and (47) it follows that

$$\pi = \sum_{\nu=0}^{m} p_\nu \mathbf{A}_\nu = \mathbf{H}\left[\delta_{ij}\lambda_j\right]_{i,j\in I}\mathbf{H}^{-1}, \tag{69}$$

where

$$\lambda_j = \sum_{\nu=0}^{m} p_\nu \lambda_{j\nu} \qquad (j=0,1,\ldots,m) \tag{70}$$

are the eigenvalues of π. Since (69) is a Jordan decomposition of π, therefore

$$\pi^n = \mathbf{H}\left[\delta_{ij}\lambda_j^n\right]_{i,j\in I}\mathbf{H}^{-1} \tag{71}$$

for all $n \geq 0$. By (57), (58), and (71) we obtain (67), and by (62) and (70) we obtain (68). This completes the proof of Theorem 7. ■

Theorem 7 shows that if we know the matrix $\mathbf{H}$ defined by (45), then we have all the information needed to determine the elements of π^n for all $n \geq 0$.

REMARKS. In the matrix $\mathbf{A}_\nu$ each row sum is equal to σ_ν, that is,

$$\sum_{j\in I} a_{ij\nu}=\sigma_\nu \tag{72}$$

for all $i\in I$. By (19) and (21) we can conclude that $\mathbf{A}_\nu$ has the following weighted column sums:

$$\sum_{i\in I} \sigma_i a_{ij\nu}=\sigma_j\sigma_\nu \tag{73}$$

for all $j\in I$ and $\nu\in I$.

Obviously, one of the eigenvalues of $\mathbf{A}_\nu$, say $\lambda_{0\nu}=\sigma_\nu$, and all the eigenvalues $\lambda_{k\nu}$ $(k\in I)$ of $\mathbf{A}_\nu$ satisfy the inequalities $|\lambda_{k\nu}|\leqslant\sigma_\nu$. We observe that the eigenvalues of the matrices $\mathbf{A}_\nu$ and $\mathbf{A}_{\nu'}$ are complex conjugate numbers, that is,

$$\lambda_{k\nu'}=\bar{\lambda}_{k\nu} \tag{74}$$

for $k\in I$ and $\nu\in I$. This follows from (56), which clearly shows that $\mathbf{\Lambda}_\nu$ and $\mathbf{\Lambda}_{\nu'}$ are complex conjugate matrices. On the other hand (51) implies that the two sets of eigenvalues $\lambda_{k\nu'}$ $(k\in I)$ and $\lambda_{k\nu}$ $(k\in I)$ contain exactly the same numbers but in different order if $\nu'\neq\nu$. If $\nu'=\nu$, then by (74) $\lambda_{k\nu}$ is necessarily real.

6. A GENERALIZATION OF GROUP CHARACTERS

Let us suppose that the conditions of Theorem 5 are satisfied, that is, G is a group of finite order, $C_0, C_1, \ldots, C_m$ satisfy Conditions 1 and 2, and $\mathbf{A}_0, \mathbf{A}_1, \ldots, \mathbf{A}_m$ commute in pairs. By Theorem 5 there exists a matrix

$$\mathbf{H}=\left[h_{ij}\right]_{i,j\in I} \tag{75}$$

such that h_{0j} $(j\in I)$ is a positive real number, and

$$\mathbf{U}=\left[h_{ij}\sigma_i^{1/2}\right]_{i,j\in I} \tag{76}$$

is a unitary matrix. In Theorem 5 the matrix (75) is uniquely determined by $\mathbf{A}_0, \mathbf{A}_1, \ldots, \mathbf{A}_m$.

Now let us define

$$\chi_{i\nu}=h_{i\nu}\sigma^{1/2} \tag{77}$$

for $i\in I$ and $\nu\in I$.

Since (76) is a unitary matrix, we have

$$\mathbf{U}\overline{\mathbf{U}}'=\overline{\mathbf{U}}'\mathbf{U}=\mathbf{I}, \tag{78}$$

and (78) implies that $\chi_{i\nu}$ $(i, \nu\in I)$ satisfy the following orthogonality relations:

$$\sum_{j=0}^{m} \chi_{ij}\bar{\chi}_{kj}=\delta_{ik}\frac{\sigma}{(\sigma_i\sigma_k)^{1/2}} \tag{79}$$

and

$$\sum_{j=0}^{m} \sigma_j\bar{\chi}_{ji}\chi_{jk}=\delta_{ik}\sigma \tag{80}$$

for $i\in I$ and $k\in I$. Moreover, (25) and (47) imply that

$$\mathbf{D}^2\mathbf{H}\mathbf{\Lambda}_\nu=\mathbf{A}'_{\nu'}\mathbf{D}^2\mathbf{H}, \tag{81}$$

and if we form the (i, k)-entry of both sides of (81), we get

$$\sigma_i\sigma_\nu\chi_{ik}\chi_{\nu k}=\chi_{0k}\sum_{j=0}^{m}\sigma_j a_{ji\nu'}\chi_{jk} \tag{82}$$

for $i, k, \nu\in I$. Here we have used (49) and (77).

We observe that if we can find real or complex numbers $\chi_{i\nu}$ $(i\in I,\ \nu\in I)$ such that (79), (80), and (82) are satisfied, and if we define $h_{i\nu}$ by (77), then all the conditions of Theorem 5 are satisfied. If we choose $\chi_{0\nu}$ $(\nu\in I)$ to be positive real numbers, then by Theorem 5 we can conclude that $\chi_{i\nu}$ $(i\in I,\ \nu\in I)$ are uniquely determined by (79), (80), and (82).

By (49) and (77) we have

$$\chi_{i\nu}=h_{i\nu}\sigma^{1/2}=\frac{\lambda_{\nu i}\chi_{0\nu}}{\sigma_i}. \tag{83}$$

If $\chi_{0\nu}$ $(\nu \in I)$ are positive real numbers, then by (74)

$$\chi_{i'\nu} = \bar{\chi}_{i\nu} \tag{84}$$

for $i \in I$ and $\nu \in I$. In addition, (51) implies that the two sets $\chi_{i'\nu}$ $(\nu \in I)$ and $\chi_{i\nu}$ $(\nu \in I)$ contain exactly the same numbers but in different order for $i' \neq i$. If $i' = i$, then by (84) $\chi_{i\nu}$ is necessarily real.

If $g \in C_i$ and $\nu \in I$, then let us write

$$\chi^{(\nu)}(g) = \chi^{(\nu)}(C_i) = \chi_{i\nu}, \tag{85}$$

where $\chi_{i\nu}$ is defined by (77). We can interpret $\chi^{(\nu)}(C_0), \chi^{(\nu)}(C_1), \ldots, \chi^{(\nu)}(C_m)$ for $\nu = 0, 1, \ldots, m$ as a generalization of the characters introduced by F. G. Frobenius [5] in 1896 for groups of finite order. See also I. Schur [8], W. Burnside [1, 2], D. E. Littlewood [7], and W. Lederman [6].

If G is a group of finite order, then the conjugacy classes of G, $C_0, C_1, \ldots, C_m$, are defined by (16) with $H = G$. The conjugacy classes $C_0, C_1, \ldots, C_m$ satisfy Conditions 1 and 2, and (43). Thus Theorem 4 is valid, and the conditions of Theorem 5 are satisfied. If in Theorem 5 we choose $h_{0\nu}$ $(\nu \in I)$ to be positive real numbers, then $\chi_{0\nu}, \chi_{1\nu}, \ldots, \chi_{m\nu}$ defined by (83) yield the characters of G. For, $\chi_{i\nu}$ $(i, \nu \in I)$ satisfy (79), (80), and (82), and these equations uniquely determine the characters of G. (See W. Burnside [2].) However, $C_0, C_1, \ldots, C_m$ need not be conjugacy classes of G for the conditions of Theorem 5 to be satisfied. Thus we can extend the notion of group characters for more general sets than conjugacy classes.

Remarks. If $C_0, C_1, \ldots, C_m$ are the conjugacy classes of G, and if $\chi^{(\nu)}(C_0), \chi^{(\nu)}(C_1), \ldots, \chi^{(\nu)}(C_m)$ $(\nu \in I)$ are the characters of G, then in Theorem 5 we can choose

$$h_{i\nu} = \chi^{(\nu)}(C_i)\sigma^{-1/2} \tag{86}$$

and

$$\lambda_{j\nu} = \sigma_\nu \frac{\chi^{(j)}(C_\nu)}{\chi^{(j)}(C_0)} \tag{87}$$

for $i, j, \nu \in I$. Evidently,

$$\sum_{j=0}^{m} \left[\chi^{(j)}(C_0)\right]^2 = \sigma, \tag{88}$$

and $\chi^{(\nu)}(C_0)$ is always a positive integer, the degree of the νth irreducible representation of G.

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